A fixed point theorem for nonexpansive compact self-mapping

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RESEARCH - PAPER

ABSTRACT. A mapping $T$ from a topological space $X$ to a topological space $Y$ is said to be compact if $T(X)$ is contained in a compact subset of $Y$. The aim of the paper is to prove the existence of fixed points of a nonexpansive compact self-mapping defined on a closed subset having a contractive jointly continuous family when the underlying space is a metric space. The proved result generalizes and extends several known results on the subject.

INTRODUCTION

Finding sufficient conditions for the existence of fixed points of a self-mapping $f$ defined on a topological space $X$ is an interesting aspect of the theory of fixed points. Several results are known in the literature on this subject. In 1930, J. Schauder proved that if $K$ is a compact convex subset of a Banach space $X$ and $T : K \rightarrow K$ is a continuous map, then $T$ has at least one fixed point in $K$. The compactness condition on $K$ is a strong one. Many problems in analysis do not have a compact setting and the spaces are not Banach spaces. So, it is natural to prove the result by relaxing the condition of compactness and considering spaces more general than Banach spaces. In this paper we also prove a result on the existence of fixed points of a nonexpansive compact self-mapping defined on a closed subset having a contractive jointly continuous family when the underlying space is a metric space.

To start with, we recall few definitions and some related results.

Let $X$ and $Y$ be topological spaces. A mapping $T : X \rightarrow Y$ is called Compact if $T(X)$ is contained in a compact subset of $Y$. 

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If \( T \) is compact, then \( \bar{T} \) is compact.

Let \((X, d)\) be a metric space. A continuous mapping \( W : X \times X \times [0, 1] \to X \) is said to be a convex structure on \( X \) if for all \( x, y \in X \) and \( t \in [0, 1], \)
\[
d(u, W(x, y, t)) \leq d(u, x) + (1 - t) d(u, y)
\]
holds for all \( u \in X \). The metric space \((X, d)\) together with a convex structure is called a convex metric space.

A subset \( K \) of a convex metric space \((X, d)\) is said to be a convex set if \( W(x, y, t) \in K \) for all \( x, y \in K \) and \( t \in [0, 1] \). The set \( K \) is said to be \( p\)-starshaped if there exists a \( p \in K \) such that \( W(x, p, t) \in K \) for all \( x \in K \) and \( t \in [0, 1] \),
\[
d(W(x, q, t), W(y, q, t)) \leq d(x, y)
\]
A normal linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces. Property (I) is always satisfied in a normal linear space.

Let \( K \) be a subset of a metric space \((X, d)\) and \( \{f_\alpha : \alpha \in K\} \) a family of functions from \([0, 1]\) into \( K \), having the property \( f_\alpha(1) = \alpha \) for each \( \alpha \in K \). Such a family is said to be
- **Contractive** if there exists a function \((0, 1) \to (0, 1)\) such that for all \( \alpha \in K \) and for \( t \in (0, 1) \), we have
\[
d(f_\alpha(t), f_\beta(t)) = d(\alpha, \beta),
\]
- **Jointly Continuous** if \( t \to t_0 \in [0, 1] \) and \( \alpha \to \alpha_0 \in K \) imply \( f_\alpha(t) \to f_\alpha(t_0) \).

In normed linear spaces these notions were discussed by Dotson.

If \( K \) is a starshaped subset with star centre \( q \) of a convex metric space \((X, d)\) with property (I), then the family \( \{f_x : x \in K\} \) defined by \( f_x(t) = W(x, q, t) \) satisfies...
\[ d(f_x(t), f_y(t)) = d(W(x, q, t), W(y, q, t) \cdot t \cdot d(x, y). \]

So, taking \((t) = t, 0 < t< 1\), the family is a contractive jointly continuous family and therefore the class of subsets of \(X\) with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

Considering the existence of fixed points for nonexpansive mappings in convex metric spaces. Beg and Abbas proved that if \(K\) is a nonempty compact and convex subset of a complete convex metric space \((X, d)\) with property \((I)\), then any nonexpansive mapping \(T : K \to K\) has a fixed point. This result is not true in general for noncompact sets \(K\). In this direction, we also prove the following.

**Theorem 1.** Let \((X, d)\) be a metric space and \(K\) a non-empty closed subset with a contractive jointly continuous family \(\{f_\alpha : \alpha \in K\}\). If \(T : K \to K\) is a nonexpansive compact mapping, then \(T\) has a fixed point.

**Proof.** Let \(K_n = , n = 1, 2, 3 \ldots \). Define \(T_n : K \to K\) as

\[ T_n x = f_{T_n} (k_n), x \in K. \]

Since \(T(K) \subseteq K\) and \(k_n < 1\), \(T_n : K \to K\) is well defined. Consider

\[ d(T_n x, T_n y) = d(f_{T_n} (k_n), f_{T_n} (k_n)) = (k_n)d(T_x, T_y) = (k_n)d(x, y), x, y \in K \]

and so each \(T_n\) is a contraction mapping on \(K\). Since \(T_n(K)\) is \(T_n\)-invariant and also compact for each \(n\) and hence complete, by Banach contraction principle, each \(T_n\) has a unique fixed point \(x_n\) in \(K\) i.e. \(T_n x_n = x_n\).

Since \(T(K)\) lies in a compact subset of \(K\), \((T x_n)\) has a subsequence such that

\[ \to x_0 \]
Now \( = () = () \rightarrow (1) = x_0 \). Since \( T \) is a continuous. \( \rightarrow Tx_0 \) and hence \( x_0 = Tx_0 \) i.e. \( x_0 \) is a fixed point of \( T \).

If \( K \) is a starshaped subset with a star centre \( p \), of a convex metric space \((X, d)\) with property (I), then the family \( \{f_\alpha : \alpha \in K\} \) defined by \( f_\alpha (t) = W(\alpha, p, t) \) is contractive if we take \( (t) = t \) for \( 0 < t < 1 \), and is jointly continuous.

**REFERENCES**

- Beg, I., Abbas, M., Fixed point and best approximation in Menger convex metric spaces, Arch. Math (Brno) 41 (2005), 389-397